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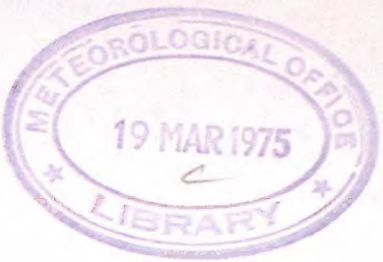

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BAROCLINIC INSTABILITY: SOME CALCULATIONS

by

B W GOLDING

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BAROCLINIC INSTABILITY: SOME CALCULATIONS

B Golding, November 1974

Introduction

Many studies of baroclinic instability have been based on Eady's pioneering work of 1949. In his paper, Eady reduced the problem of the stability of a differentially heated atmosphere on a rotating Earth, to its simplest possible state while retaining its essential characteristics. Thus he replaced the Earth by a flat plate rotating about its centre, and capped his atmosphere with a rigid lid. He also assumed a quasi-static, locally incompressible atmosphere. Under these restrictions, he obtained an expression for the growth rate of a sinusoidal perturbation on a constant zonal flow. In particular, he found that, for instability, the wavelength of the perturbation could not be shorter than a critical value.

Since 1949, many papers have been published extending Eady's work by relaxing or changing some of his constraints. The two changes of most interest to me in this paper are

(i) the introduction of a potential vorticity gradient (PVG) and

(ii) the introduction of sloping and viscous boundaries.

Two forms of PVG may be introduced in a straightforward manner. Firstly the variation of Coriolis parameter with latitude, assumed constant and denoted β. Slightly less simple is the case where the vertical wind profile is not linear. Green (1962) produced a numerical solution to these equations which showed the absence of a short wave limit to instability. Using a value of β suitable for the earth, his solutions showed high growth rates in the short wave regime with the longest waves being almost stable. A direct solution of these equations by Simmons (1974), hereafter referred to as the series method, forms the basis of part of this paper. The other method of introducing a PVG, which I shall use, is due to McIntyre (1970), and will hereafter be referred to as the perturbation method. It is more general in that the mean flow can vary in the y direction. However, it is restrictive

in that the PVG is treated as a perturbation and must therefore be small. McIntyre's solution also removed the short wave limit but the perturbation values of β failed to affect the long waves.

The introduction of viscous and sloping boundaries in the problem involves a change in the conditions at the upper and lower boundaries. Hide (1969) introduced this change into Eady's solution. His solution indicates that there will be a long wavelength limit to instability when the boundaries slope in the same sense as the isotherms. When viscosity is introduced in the form of Ekman boundary layers on both surfaces, his solutions show a reduction in growth rate of all wavelengths. If the upper surface is left inviscid, however, the long and short stable wavelengths are destabilised.

In this paper I shall introduce sloping and viscous boundaries into both perturbation and series methods. I hope to show that the effects are similar to those found in Hide's solution depending on the relative importance of the destabilising PVG and the stabilising viscosity. I shall also show the effect of typical values of β and turbulent viscosity for the Earth.

2 The Equations of Motion

The system under consideration is shown in Fig 1. The atmosphere has been represented on a beta-plane with rigid boundaries in the y and z directions. The upper and lower boundaries are permitted to slope at small angles from the horizontal. The fluid is inviscid except in Ekman layers at the upper and lower boundaries. The axis of rotation is assumed parallel to the z axis and gravity antiparallel to it. Horizontal distances have been non-dimensionalised with respect to the channel width and vertical distances with respect to its height.

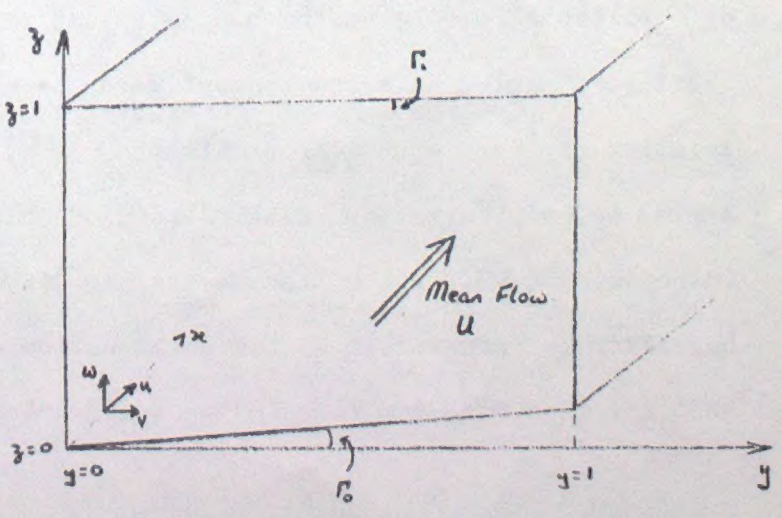


Fig 1

The system under consideration. The velocity field is $\underline{u} = (u,v,\omega)$

The fluid is assumed to obey the Boussinesq approximation and to be locally incompressible. The non-dimensional equations of motion for a stratified fluid subject to these conditions may be written (Greenspan 1968)

$$\nabla \cdot \underline{u} = 0 \qquad \text{Continuity} \qquad (2.1)$$

$$R \frac{D\underline{u}}{Dt} + \frac{f}{f_0} \hat{\underline{k}} \wedge \underline{u} = -\nabla p - \rho \hat{\underline{k}} \qquad \text{Momentum} \qquad (2.2)$$

$$\omega - R\epsilon \frac{D\rho}{Dt} = 0 \qquad \text{Thermodynamic} \qquad (2.3)$$

where $\epsilon = \frac{-f_0^2 L}{\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial y}}$ is formally of order one and $R = \frac{U}{f_0 L}$ a Rossby number for the mean flow, is assumed small. p, ρ are deviations, from the mean, of pressure and density. In the definition of ϵ, ρ_0 is a mean density, and $\bar{\rho}(y)$ a horizontally meaned density.

$\frac{D}{Dt}$ denotes the total derivative following the flow. f_0 is a mean of $f = f(y)$, the Coriolis parameter. U and L are scales for the horizontal velocity and length respectively.

The dependent variables are expanded in powers of R. The zeroth order solution then gives the geostrophic relation

$$\hat{\underline{k}} \wedge \underline{u}_0 = -\nabla p_0 \qquad (2.4)$$

which defines p_0 as a stream function for the horizontal velocity field, and the hydrostatic equation

$$\frac{\partial p_0}{\partial y} = -\rho_0 \qquad (2.5)$$

The first order equations may be combined to yield an equation in the zeroth order vorticity, $\zeta = \nabla^2 p_0 + \frac{f}{R f_0}$

$$\frac{D\zeta}{Dt} = \frac{1}{R}\frac{\partial \omega_1}{\partial y} \tag{2.6}$$

and an equation for the vertical velocity

$$-\frac{\omega_1}{R} = \frac{D}{Dt}\left(\varepsilon \frac{\partial \rho_0}{\partial y}\right) \tag{2.7}$$

These may be combined to a relationship between zeroth order quantities:

$$\frac{D}{Dt}\left(\zeta + \varepsilon \frac{\partial \rho_0}{\partial y}\right) = 0 \tag{2.8}$$

The boundaries at $y = 0, 1$ are assumed to be inviscid. This gives the zero normal velocity condition

$$v = 0 \quad \text{at} \quad y = 0, 1 \tag{2.9}$$

For the conditions at $y = 0, 1$, an investigation of the properties of the viscous boundary layers is required. They are Ekman layers of thickness $O(\varepsilon^{1/2})$ depending on a balance between Coriolis and viscous forces. The equations may be solved analytically for boundaries sloping at small angles Γ_i $(i = 0, 1)$ to the horizontal, and yield a condition on the interior vertical velocity. The physical consequence is an $O(\varepsilon^{1/2})$ vertical velocity in the interior given by the equations:

$$\omega_1 = \Gamma_0 v_0 + \frac{E_0^{1/2}}{\sqrt{2}}\zeta \quad \text{at } y = 0$$

$$\omega_1 = \Gamma_1 v_0 - \frac{E_1^{1/2}}{\sqrt{2}}\zeta \quad \text{at } y = 1 \tag{2.10}$$

where $E_i = \frac{\nu_i}{\Omega_0 L^2}$ are Ekman numbers appropriate to the boundary layer flow at each surface. The eigenvalue problem for normal mode disturbances to the mean flow $U(y,z)$ is obtained by posing

$$\rho_0 = -\int^y U(\eta,z)\,d\eta + \psi' \quad : \quad \psi' = Re\left\{\phi(y,z)\,e^{ik(x-ct)}\right\} \tag{2.11}$$

where $|\psi'| \ll |\rho_0|$. Substituting in (2.8) and linearising gives

$$(U-c)\left[\phi_{yy} + \varepsilon\phi_{zz} - k^2\phi\right] + \left(\frac{1}{R\Omega_0}\frac{\partial \Omega}{\partial y} - U_{yy} - \varepsilon U_{zz}\right)\phi = 0 \tag{2.12}$$

Now set $\beta = \frac{1}{R\Omega_0}\frac{\partial \Omega}{\partial y}$ (2.13) and $q_y = \beta - U_{yy} - \varepsilon U_{zz}$ (2.14), the gradient of potential vorticity in the y direction.

Similarly, substituting (2.11) in the boundary conditions and linearising gives

$$\begin{aligned}
&(U-c)\phi_z - \left(U_z - \frac{\Gamma_0}{R\varepsilon} - \frac{iE_0^{1/2}k}{\sqrt{2}R\varepsilon}\right)\phi - \frac{iE_0^{1/2}}{\sqrt{2}kR\varepsilon}\phi_{yy} = 0 \\
&(U-c)\phi_z - \left(U_z - \frac{\Gamma_1}{R\varepsilon} + \frac{iE_1^{1/2}k}{\sqrt{2}R\varepsilon}\right)\phi + \frac{iE_1^{1/2}}{\sqrt{2}kR\varepsilon}\phi_{yy} = 0
\end{aligned} \tag{2.15}$$

$$\phi = 0 \quad \text{at} \quad y = 0,1 \tag{2.16}$$

Solution of the Eigenvalue Problem

The two methods of solution must be treated separately at this point.

(i) The perturbation method.

The perturbation series is set up by posing

$$U = z + \mu u_1(y,z) \tag{3.1}$$

and

$$q_y = \mu q_{1,y} = \mu\beta_1 - \mu u_{1,yy} - \mu\varepsilon u_{1,zz} \tag{3.2}$$

where μ is small. Now, expanding ϕ as

$$\phi = \phi_0 + \mu\phi_1 + \mu^2\phi_2 \tag{3.3}$$

and treating c as an eigenvalue with the other parameters fixed

$$c = c_0 + \mu c_1 + \mu^2 c_2 \tag{3.4}$$

completes the perturbation formalism.

The solution for ϕ_1, c_1 follows exactly the procedure used by McIntyre. The zero order solution (ϕ_0, c_0) is identical to that of Hide (1969). As in McIntyre's solution, the first correction only is computed, and this may be determined directly from the zero order solution. In the light of the zero order result, the friction parameters F_u, F_l are defined by

$$F_l = \frac{E_0^{1/2}\gamma^2}{\sqrt{2}Rk} \quad , \quad F_u = +\frac{E_1^{1/2}\gamma^2}{\sqrt{2}Rk} \tag{3.5}$$

where γ is the total wavenumber

$$\gamma = \sqrt{\frac{k^2+m^2\pi^2}{\varepsilon}} \tag{3.6}$$

and m is a wavenumber in the y direction.

(ii) The series method.

A number of simplifications are made to (2.12) before proceeding. Firstly, the mean velocity U is assumed to have the simple form $U = z$. Also, $\phi(y,z)$ is assumed to have the form $\phi(y,z) = \phi(z)\sin my$ (3.7) Defining γ and F as above (2.12) and (2.15) may be written

$$(z-c)[\phi_{zz} - \gamma^2\phi] + \frac{\beta}{\varepsilon}\phi = 0 \tag{3.8}$$

$$(z-c)\phi_z - (1 - Q_L - iF_L)\phi = 0 \quad \text{on } z = 0$$

$$(z-c)\phi_z - (1 - Q_u + iF_u)\phi = 0 \quad \text{on } z = 1 \qquad (3.9)$$

$$\text{where } Q_L = \frac{\Gamma_0}{R\varepsilon} \quad \text{and} \quad Q_u = \frac{\Gamma_1}{R\varepsilon}$$

Note that if $\beta = 0$, equation (3.8) reduces to a simple wave equation which may be solved directly. This was the equation derived by Eady (1949).

This equation may be manipulated into a form of Whittaker's Equn, having the solutions given in Abramowitz and Stegun (13.1). An expression for ϕ_z/ϕ can then be derived and using the values given in (3.9) an equation for the eigenvalue c is deduced. Since the solutions to Whittaker's Equn are infinite series, the expression for c is an implicit one requiring iterative solution.

Results

Using McIntyre's perturbation technique, results were computed for cases:

(a1) Perturbed velocity profile $U = z_s - 2z_s y_s^2$, $\beta = 0$, where $z_s = z - 0.25$, $y_s = y - 0.5$

(a2) Unperturbed velocity profile $U = z$, $\beta = 0.316$

Using Simmons' series solution, results were computed for cases:

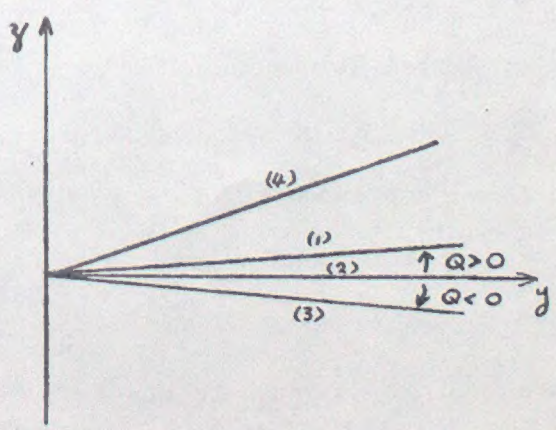
(b1) $\beta = 0.316$

(b2) $\beta = 10.0$

In all cases the slopes used were combinations of $Q_u, Q_L = 0.5, 0.0, -0.5$ as shown in Fig 2. The viscosities were $F_u, F_L = 0.1, 0.01, 0.0$, and the stability parameter ε was set at 10 throughout.

Fig 2

The directions of the slopes used in calculations. Values were (1) $Q = 0.5$ (2) $Q = 0.0$ (3) $Q = -0.5$. Also shown is (4) the isotherm slope, $Q = 1.0$.



The overlapping cases (a2), (b1) were run in order to check the agreement of the two methods. A comparison is shown in Fig 4 for the case $Q_L = Q_u = 0.5$.

There is qualitative agreement, but the perturbation method has produced larger changes from the unperturbed solution.

The solutions of Eady (1949) and Hide (1969), with no potential vorticity gradient, show a region of instability where position on the wavenumber axis depends on the slope of the boundaries, as shown in Fig 3. With either slope in position (3) - see Fig 2 - a long wave cut-off to instability occurs. With all slopes, a short wave cut-off exists. With viscosity, the growth rate of this waveband is decreased, but the wavelength of each cut-off is only slightly affected.

The solutions with $\beta = 0.316$ show a destabilisation of the short wave regime, and very little effect elsewhere. In particular, the long wave stable regime of Fig 3 (3.4) remains close to neutral stability. A typical solution is shown in Fig 4. Others are not reproduced here since the prime interest lies in the remaining two cases.

The parabolic velocity profile introduced in (a1) represents a typical profile found in the rotating annulus experiments. The effects of introducing this perturbation are shown in Figs 5-7. Fig 5 shows the perturbed Eady solution. In Fig 6, the boundaries slope in opposite directions while Fig 7 has the boundaries sloping away from the isotherms. Short wave destabilisation occurs in all cases, but the cut-off still exists at a varying distance from the unperturbed cut-off. Where the boundary slope is in position 3, ie Fig 7, the long wave regime is slightly destabilised. Growth rates in the main growing regime are reduced a little by the perturbation. Note that the destabilised short waves travel slower than the mean fluid speed $U = 0.25$, ie they are steered by a level below $z = 0.5$. However, the long waves travel faster than the mean speed corresponding to a high steering level. It should be noted that if the level of zero mean velocity approaches $z = 0$ or $z = 0.5$, the destabilisation vanishes. The case where $U = 0$ at $z = 0.5$ is discussed in McIntyre (1970). The case where $U = 0$ at $z = 0$ can be understood in terms of the potential vorticity argument of Bretherton (1966a, b). When viscosity is introduced on both boundaries, growth rates are generally reduced, leading to the short and long wave regimes becoming stable. When viscosity is applied to the lower surface

only, it has the same effect in the perturbed solutions. However, in the unperturbed case, the short and long wave regimes are destabilised as shown in Fig 8 a) and b). Fig 8a) is the modified Eady solution.

The solutions obtained with the series method (b2) used magnitudes for the parameters which are similar to those found in the atmosphere. For the horizontal case, this gives the curve shown in Fig 9. A number of features are clear from this result. Firstly, the reduction in growth rate of the long waves. This is accompanied by a shift to shorter wavelengths of the fastest growing wave. Note also the high growth rates of the short waves. Another change, is the considerable decrease in the phase velocity of the waves relative to the mean flow. The highest phase speed shown corresponds to a steering level at about $z = 0.3$, well below the middle of the fluid. Fig 10 shows that the long wave regime is also markedly destabilised in the case of parallel boundaries sloping contrary to the isotherms.

When the approximation of modelling the Earth's boundary layer as an Ekman layer is made, the coefficient of viscosity used must be a turbulent one, many times larger than the equivalent laminar flow viscosity. The large value of $F = 0.1$ was used for this. It should be noted that F is not a true viscosity since it varies with the wavenumber. The result can be seen in Fig 11. Although the short waves are not completely damped out, the growth rates are considerably reduced. Note, however, that the wavenumber of maximum growth rate is virtually unchanged.

To investigate more closely the effects of the Earth's boundary layer, I used Dr Simmons' original program, modified only to include viscous boundaries. This model includes effects of compressibility by introducing a scale density height. It also uses dimensional input parameters. Two cases are displayed in Fig 12. The first has no friction and is essentially a repeat of Green's solution (1960). In this solution, it is seen that high growth rates are maintained to wavenumber $4 \times 10^{-6}\ m^{-1}$ and clearly extend much higher. However in Fig 12b) the solution is modified to take account of the turbulent viscosity at the Earth's surface. The upper surface is left inviscid although the result

is almost identical if it is not. The value of kinematic viscosity is $2\ m^2 s^{-1}$. Note that the parameter used in this formulation is a true viscosity and so the frictional effect caused by it increases with increasing wavenumber. It is seen that growth rates are sharply cut back in the short wave regime. It is interesting to note, however, that the most unstable wavenumber is almost unchanged after the introduction of viscosity.

To keep consistency with the results described above, Fig 12 was produced for an effectively incompressible atmosphere. To illustrate the effect of compressibility, Fig 13 uses a scale density height of 8 Km - a suitable value for the Earth. The growth rates of the short waves are higher than in the incompressible case, but the effect of introducing viscosity is similar, resulting in a cut-off at about $4 \times 10^{-6}\ m^{-1}$.

Conclusions

For the instability of a baroclinic fluid in a rotating annulus, it has been shown that slight short wave destabilisation occurs for all boundary slope combinations when a parabolic velocity profile is introduced. There is slight destabilisation in the long wave regime only under certain conditions. The stabilising effect of viscosity was also noted.

The results using atmospheric magnitudes for the variable parameters, show very unstable short waves and a shift of the maximum growth rate as in Green's solution. The destabilisation of the long waves has also been shown to be considerable in the case of suitably sloping boundaries. The growth rates of the short waves are considerably greater than has been suggested by some writers. Since they correspond to the very shortest synoptic waves observed in the atmosphere, they would be a serious problem to numerical forecasters if real. However, it has also been shown that introducing a suitable value of turbulent viscosity leads to these growth rates being considerably reduced.

Acknowledgements

I should particularly like to thank Dr P J Mason who introduced me to the subject and who has been a constant source of help and encouragement. I am also indebted to Dr A J Simmons of Reading University for allowing me to use his computer programmes. The opportunity to discuss parts of the work with Dr McIntyre was greatly appreciated.

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Legend

Fig 3 The results of Hide and Eady with no potential vorticity gradient. Case (1) with $Q_u = Q_l = 0.0$ is Eady's solution.

Fig 4 A comparison of the perturbation and series methods with a value of β appropriate to the PVG in a rotating annulus, and with parallel boundaries sloping towards the isotherms. Compare with Fig 3 (2).

Fig 5 The effects on Eady's solution of introducing a parabolic velocity profile. This should be compared with Fig 3 (1).

Fig 6 Here, the boundaries are sloping in opposite directions as in Fig 3 (4).

Fig 7 The boundaries are parallel and sloping contrary to the isotherms. Compare with Fig 3 (3).

Fig 8 Viscosity is applied to the lower boundary only, with no potential vorticity gradient. In a) the boundaries are horizontal, corresponding to the Eady solution. In b) the boundaries are parallel and sloping contrary to the isotherms.

Fig 9 The effects on Eady's solution of introducing a β term comparable with that on the Earth. Compare with Fig 3 (1).

Fig 10 The conditions are the same as those in Fig 9 except that friction has been introduced on the boundaries.

Fig 11 The boundaries are parallel and sloping away from the isotherms. No friction is applied in this case. Compare with Fig 3 (3).

Fig 12 This solution was produced by Dr Simmons' modified program. The dimensions of the parameters are shown on the graph. The value of scale density height used here corresponds to an effectively incompressible atmosphere. The result shows the reduction in growth rate produced by the introduction of a turbulent viscosity similar to that found on the earth.

Fig 13 The conditions are the same as those in Fig 12 except that a realistic scale density height, for the Earth's atmosphere, has been used. Note the shift of origin which cuts off the long wave regime. This solution shows the effect of introducing a turbulent viscosity when parameters suitable to consideration of baroclinic instability on the Earth are used.

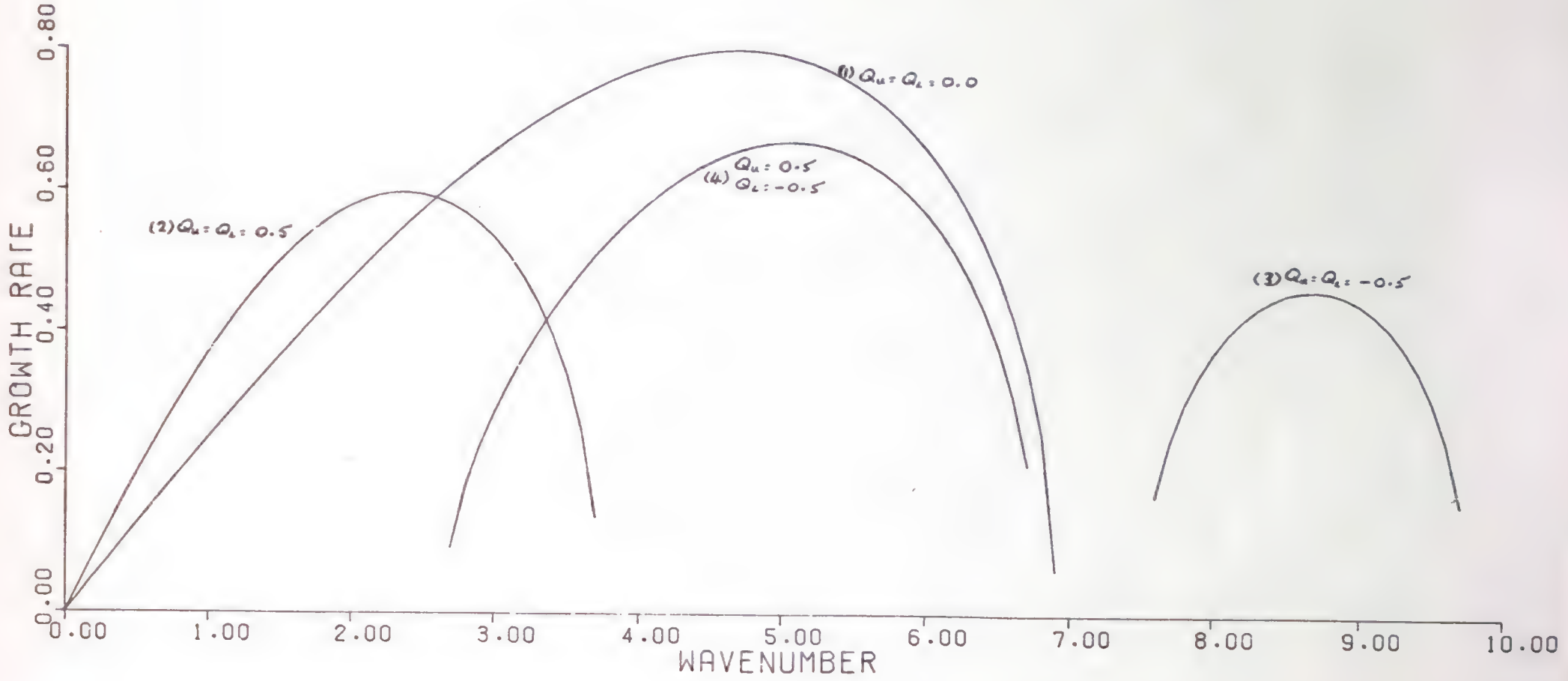
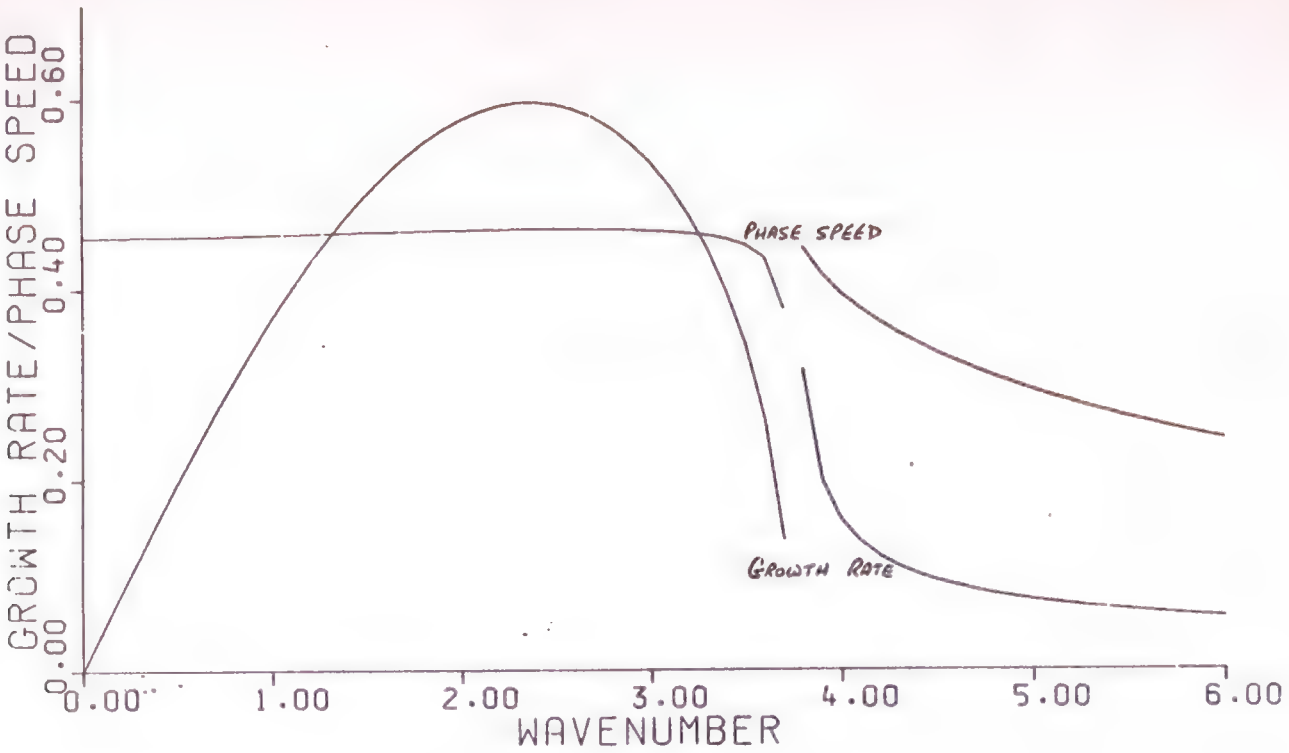


Fig 3 $\beta = 0$

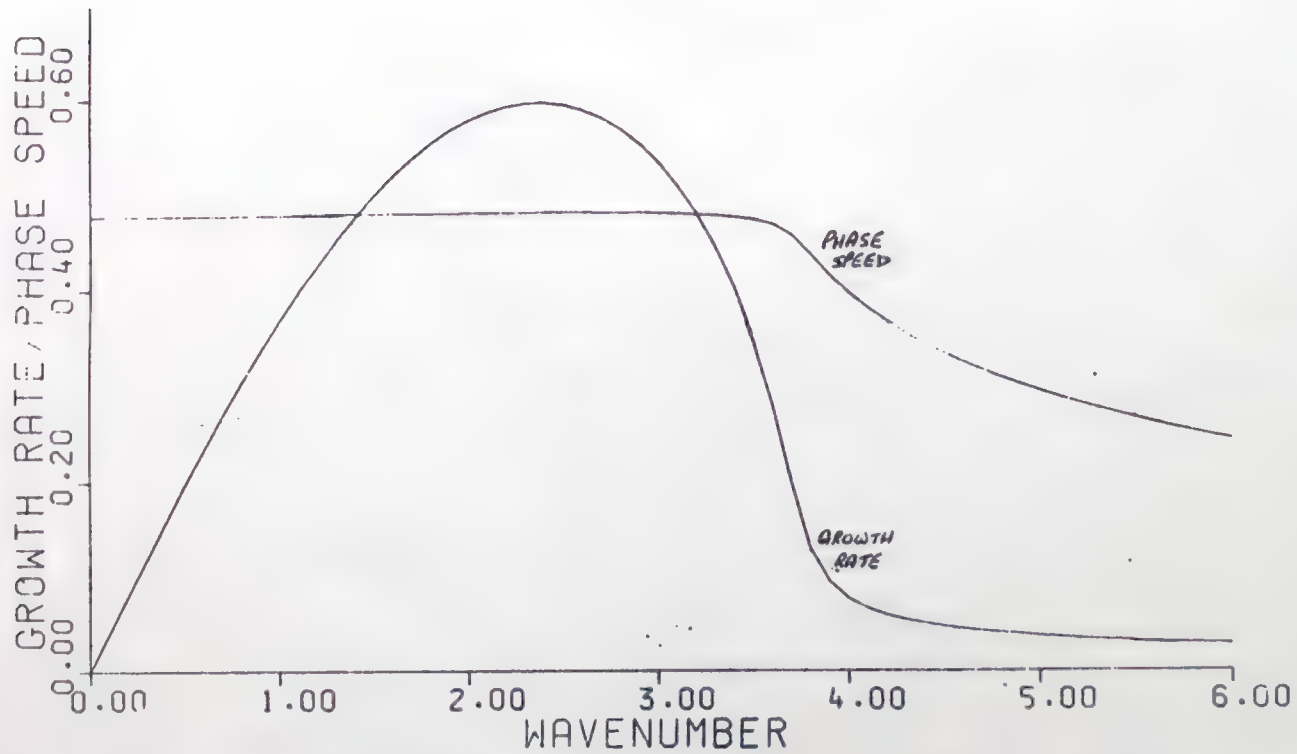
$F_u = F_L = 0$



PERTURBATION METHOD $\beta = 0.316$

$Q_u = Q_L = 0.5$

$F_u = F_L = 0.0$



SERIES METHOD $\beta = 0.316$

$Q_u = Q_L = 0.5$

$F_u = F_L = 0.0$

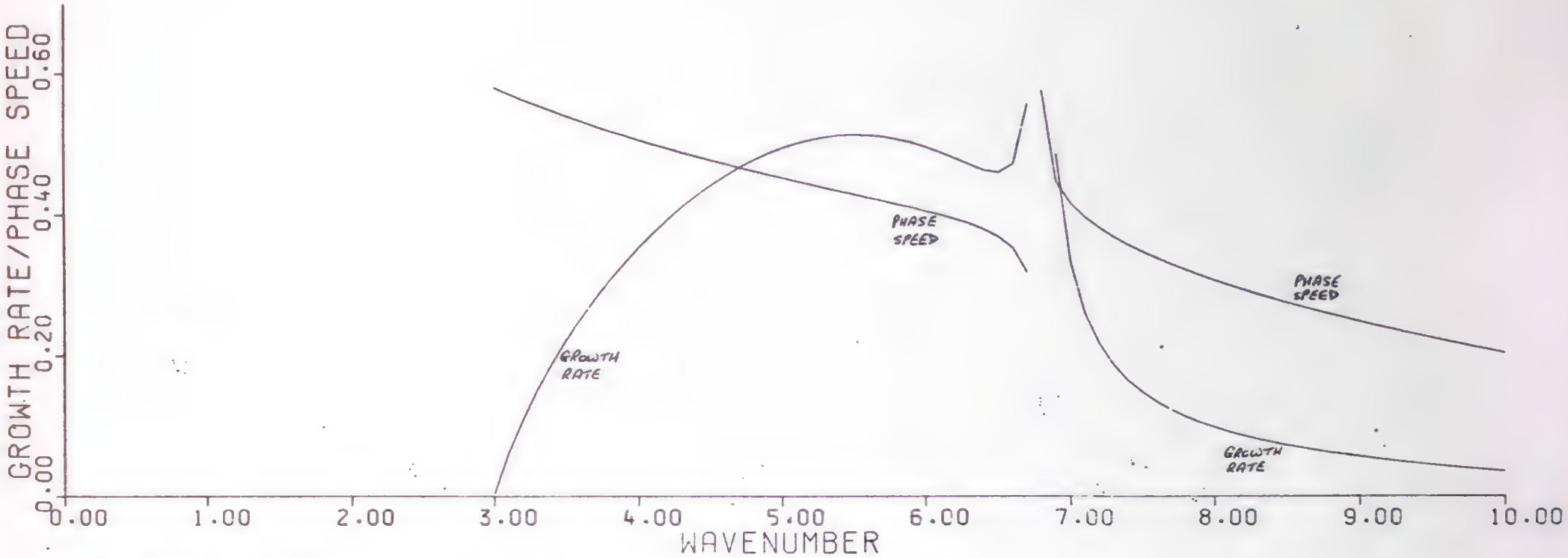
Fig 4

PERTURBATION METHOD PARABOLIC VELOCITY PROFILE

$Q_u = Q_l = 0.0$

$F_u = F_l = 0.0$

Fig 5

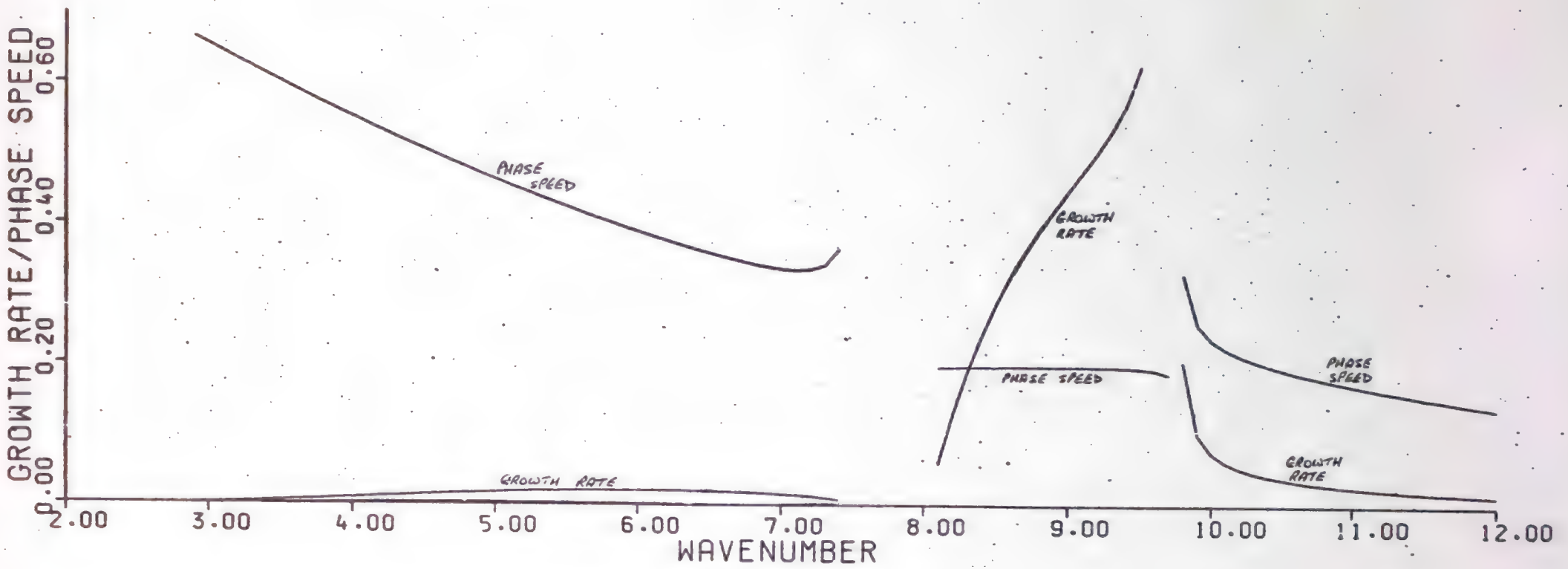


PERTURBATION METHOD PARABOLIC VELOCITY PROFILE

$Q_u = 0.5$ $Q_L = -0.5$

Fig 6

$F_u = F_L = 0.0$



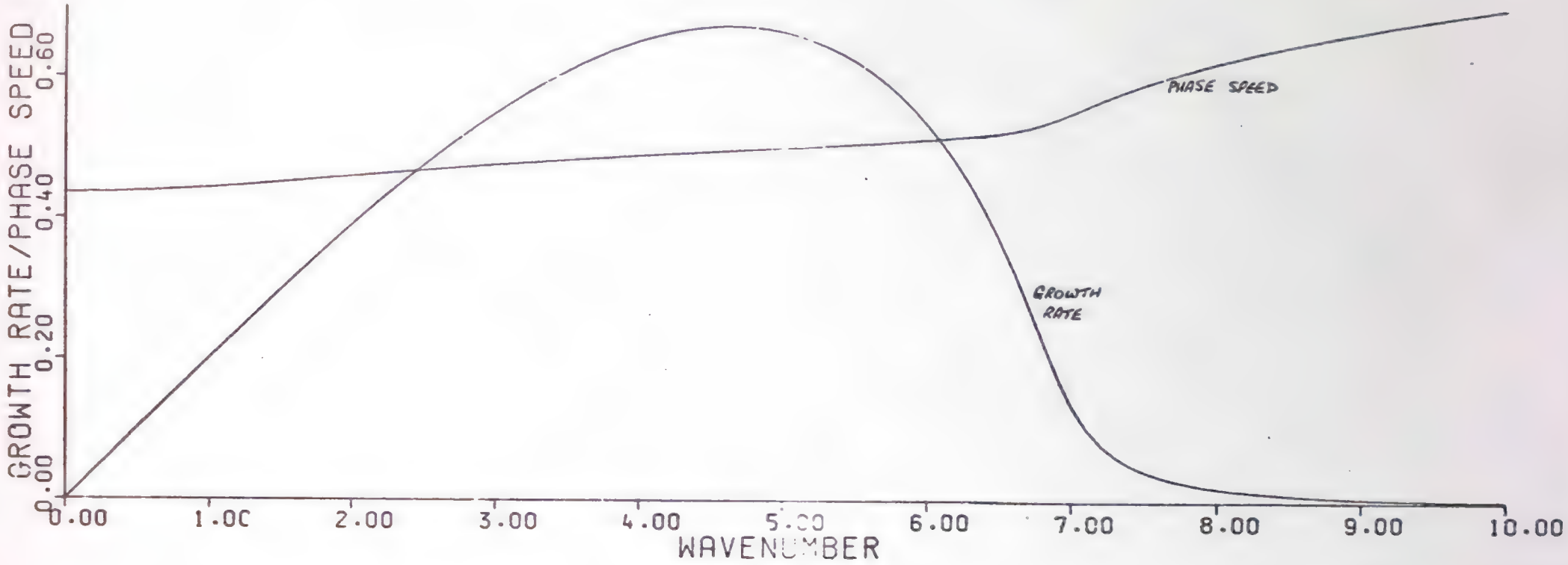
PERTURBATION METHOD

PARABOLIC VELOCITY PROFILE

$Q_u = Q_L = -0.5$

$F_u = F_L = 0.0$

Fig 7

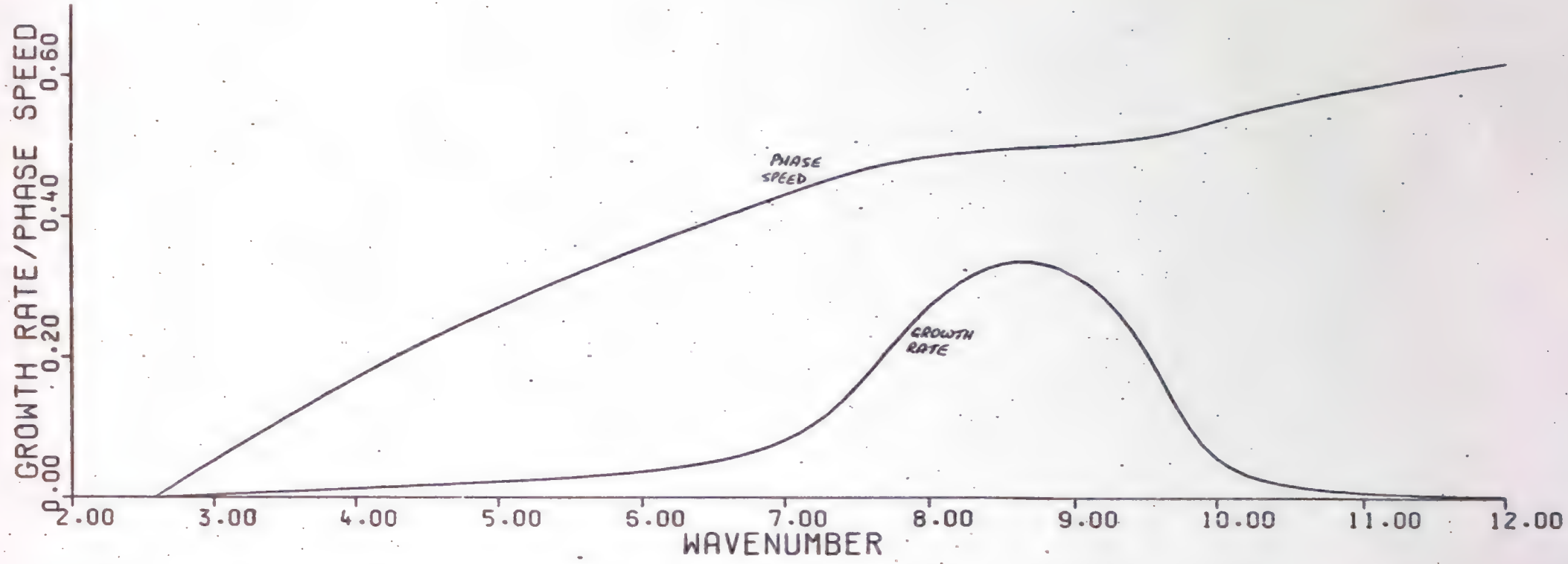


No PVG

$Q_u = Q_L = 0.0$

$F_u = 0.0 \quad F_L = 0.1$

$\underline{F_y}\ \underline{\beta_u}$

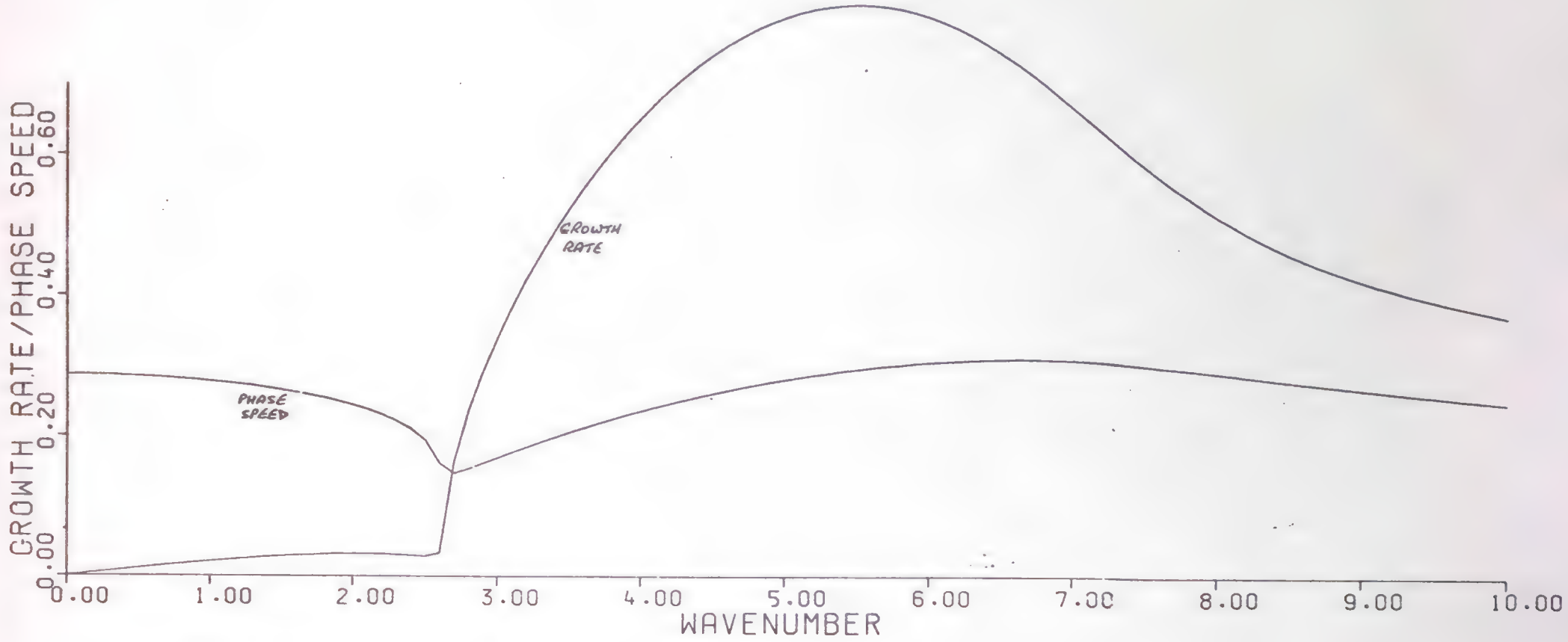


NO PVG

$Q_u = Q_L = -0.5$

$F_u = 0.0 \quad F_L = 0.1$

Fig 8b)

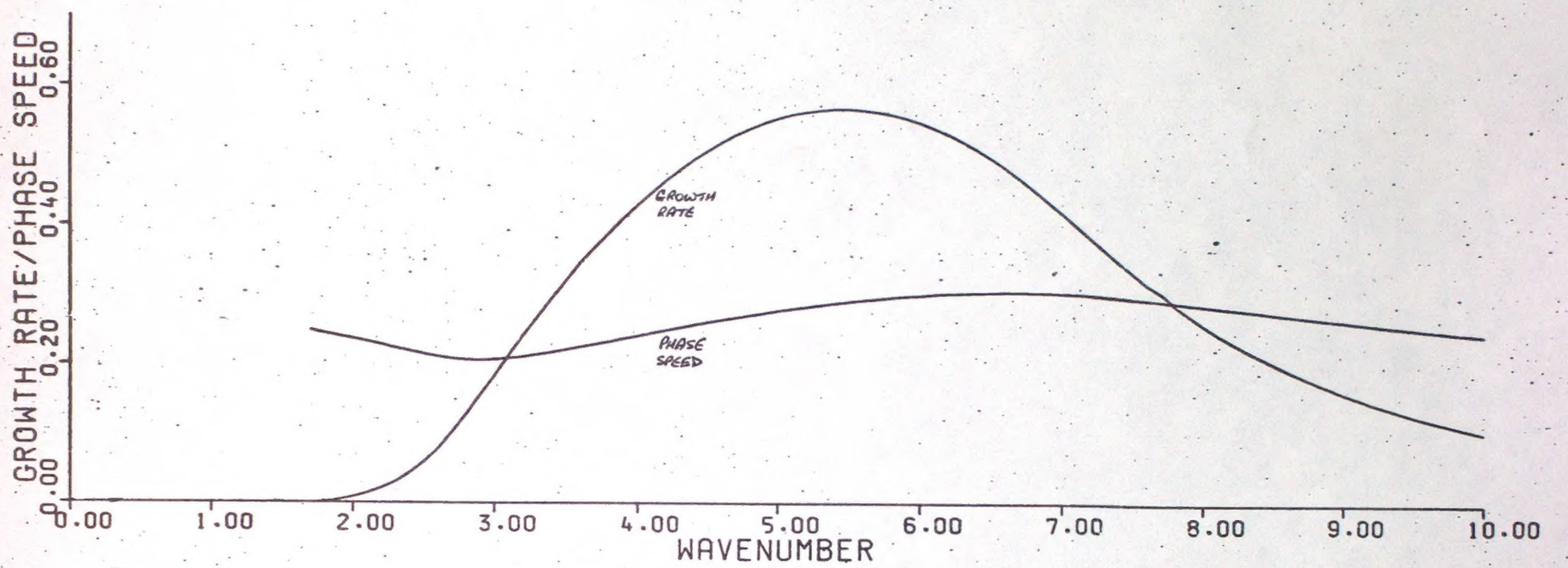


SERIES METHOD $\beta = 10.0$

$Q_u = Q_L = 0.0$

$F_u = F_L = 0.0$

Fig 9

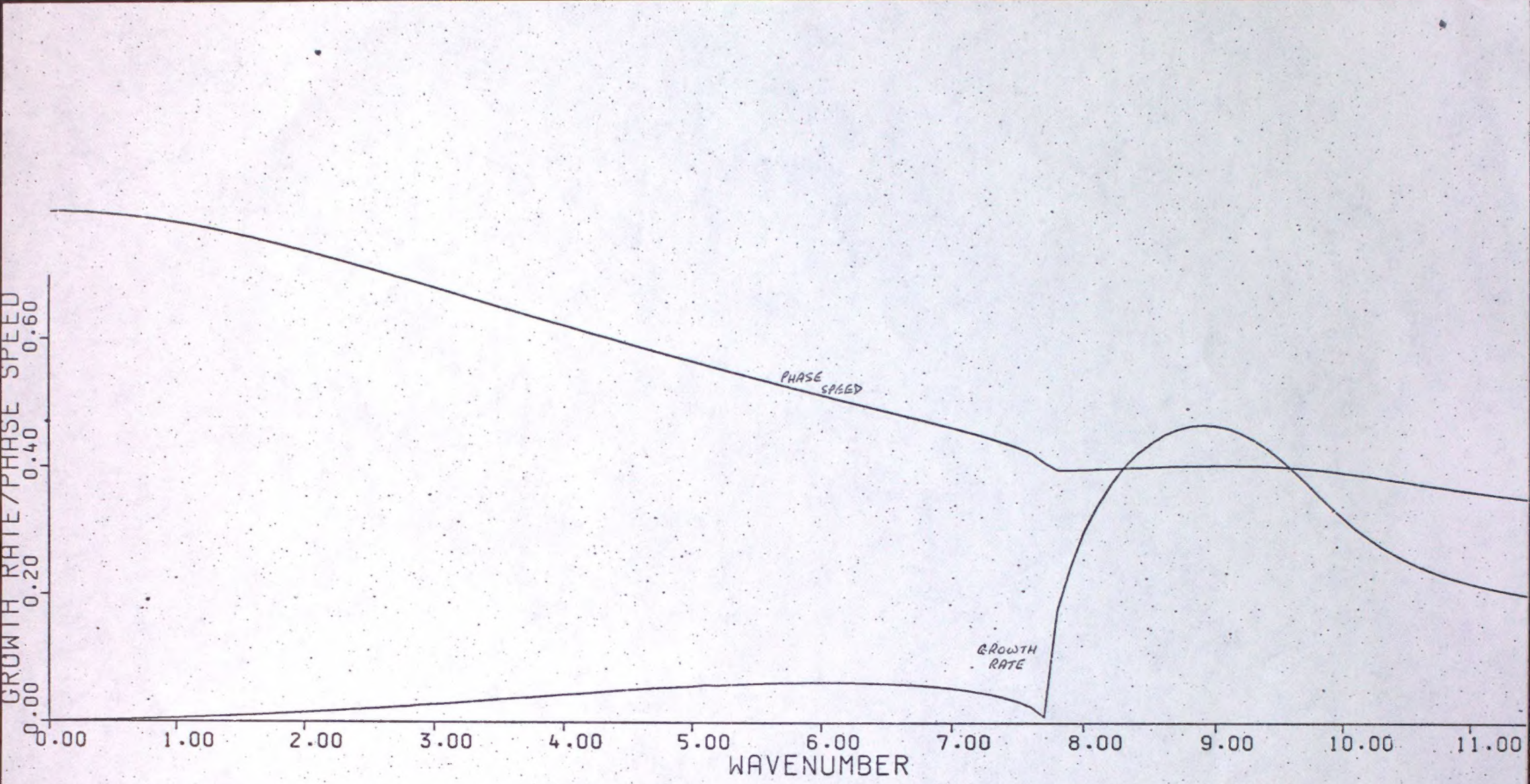


SERIES METHOD $\beta = 10.0$

$Q_u = Q_L = 0.0$

$F_u = F_L = 0.1$

Fig 10



SERIES METHOD $\beta = 10.0$

$Q_u = Q_L = -0.5$

$F_u = F_L = 0.0$

Fig 11

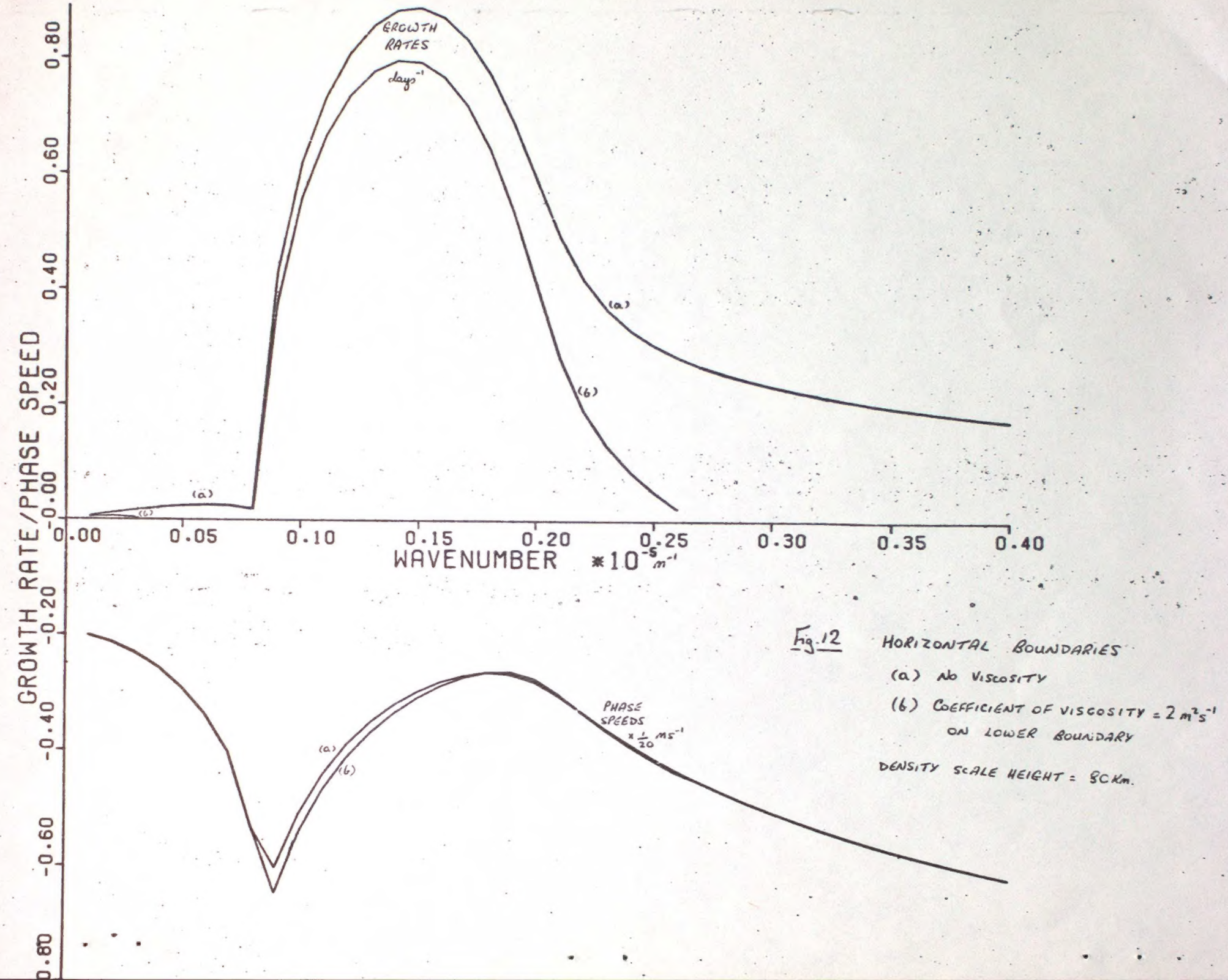


Fig. 12 HORIZONTAL BOUNDARIES

(a) No Viscosity

(b) Coefficient of viscosity = $2\ m^2 s^{-1}$ on lower boundary

DENSITY SCALE HEIGHT = 8C Km.

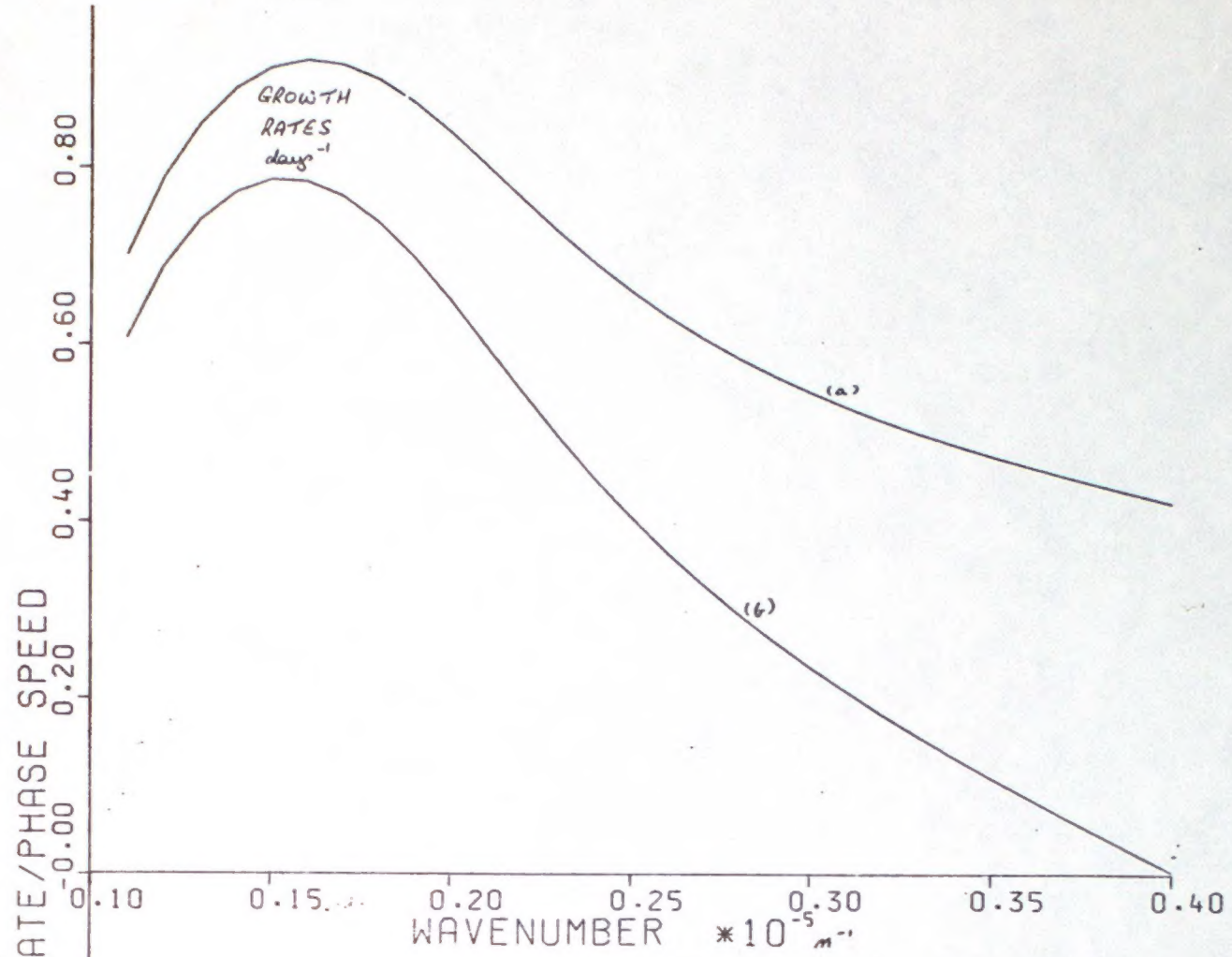


Fig 13 HORIZONTAL BOUNDARIES

(a) NO VISCOSITY

(b) COEFFICIENT OF VISCOSITY = $2\ m^2 s^{-1}$ ON LOWER BOUNDARY

DENSITY SCALE HEIGHT = 8Km